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Simultaneous Uniformization

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We shall show that any <u>two</u> Riemann surfaces satisfying a certain condition, for instance, any two closed surfaces of the same genus $g > 1$, can be uniformized by <u>one</u> group of fractional linear transformations (Theorem 1). This leads, in conjunction with previous results [2,3], to the simultaneous uniformization of <u>all</u> algebraic curves of a given genus (Theorems 2-4). Theorem 5 contains an application to infinitely dimensional Teichmüller spaces.

<u>1</u>. Let S be an abstract Riemann surface, f a homeomorphism of bounded eccentricity of S onto another such surface S', and $[f]$ the homotopy class of f. We call $(S,[f],S')$ a <u>coupled pair</u> of Riemann surfaces, an <u>even</u> (<u>odd</u>) pair if f preserves (reverses) orientation. Two coupled pairs, $(S,[f],S')$ and $(S_1, [f_1], S_1')$ are called <u>equivalent</u> if there exist conformal homeomorphisms h and h' with $h(S) = S_1$, $h'(S) = S_1'$, and $[h'fh^{-1}] = [f_1]$.

Example. Let m be a Beltrami differential on the Riemann surface S_o, i.e. a differential of type $(-1,1)$, $m = (\zeta)d\bar{\zeta}/d\zeta$, with $|\mu| \leq$ const. < 1. By S_o^m we denote the surface S_o with the conformal structure redefined by means of the local metric $|d\zeta+\mu d\bar{\zeta}|$. With m there is associated the even pair $(S_o^m,[1],S)$, where 1 is the identity mapping, and the odd pair $(S_o^m,[\iota],\bar{S}_o)$ where ι denotes the natural mapping of S_o onto its mirror image $\bar{S}_o$. The latter is defined by replacing each local uniformization ζ on S_o by $\bar{\zeta}$.

A group G of Mobius transformations will be called <u>quasi-Fuchsian</u> if there exists an oriented Jordan curve γ_G (on the Riemann sphere P) which is fixed under G, and if G is fixed-point-free and properly discontinuous in the domains I (γ_G) and E (γ_G) interior and exterior to γ_G, respectively. If γ_G is a circle, G is a Fuchsian group.

A quasi-Fuchisian group G is canonically isomorphorphic to the fundamental groups of the two Riemann surfaces $S_1 = I(\gamma_G)/G$ and $S_2 = E(\gamma_G)/G$, modulo inner sutomorphisms. If the resulting isomorphisms of the fundamental groups of S_1 onto those of S_2 can be induced by an orientation reversing homeomorphism f of bounded eccentricity, G is called <u>proper</u>. In this case [f] is uniquely determined. Thus a proper quasi-Fuchsian group represents a coupled pair $(S_1,[f],S_2)$.

A quasi-Fuchsian group G is said to be of the <u>first</u> (<u>second</u>) <u>kind</u> if the fixed points of elements of G are (are not) dense or γ_G. This is, as one sees at once, a property of S_1 (or of S_2).

<u>2</u>. <u>Theorem 1</u>. <u>Let</u> S <u>be a Riemann surface with hyperbolic universal covering surface and</u> (S,[f],S') <u>an odd coupled pair. Then this pair</u> (<u>is equivalent to one which</u>) <u>can be represented by a quasi-Fuchisian group</u> G. <u>If</u> G <u>is of the first kind, then every quasi-Fuchisan group</u> G_1 <u>representing an equivalent coupled pair is of the form</u> $G_1 = CGC^{-1}$ <u>where</u> C <u>is a Möbius transformation.</u>

Proof. One sees easily that any odd coupled pair is equivalent to one of the form $(S_o^m, [\iota], \overline{S}_o)$; we assume therefore that the given pair already has this form. If S_o^m is not the sphere, the plane, the punctured plane or a torus, the same is true of S_o. In this case the classical uniformization theorem asserts

that the pair $(S_o, [\ell], \overline{S}_o)$ can be represented by a Fuchsian group G_o; we may assume that γ_{G_o} is the real axis. There exists a measurable function $\mu(z)$, $|z| < \infty$, such that $\mu(z) \equiv 0$ for Im $z \leq 0$ and $\mu(z)\, d\overline{z}/dz = m$ for Im $z > 0$. Then $|\mu| \leq$ const. <1 and $\mu(z)d\overline{z}/dz$ is invariant under G_o. It is known (cf. for instance, [1]) that there exists a unique solution $\Omega_m(z)$ of the Beltrami equation $\partial\Omega/\partial\overline{z} = \mu(z)\partial\Omega/\partial z$ which has generalized L_2 derivatives and is a homeomorphism of P onto itself, with fixed points at 0, 1, ∞. If $A_o \varepsilon G_o$, then $\Omega_m A_o$ satisfies the same Beltrami equation; it follows that there is a Möbius transformation A with $\Omega_m A_o = A\Omega_m$. One verifies easily that $G = \Omega_m G_o \Omega_m^{-1}$ is a quasi-Fuchsian group representing $(S_o^m, [\ell], \overline{S}_o)$. We note that $\gamma_G = \Omega_m(\gamma_{G_o})$ has two-dimensional measure zero.

Assume next that G is of the first kind and that the quasi-Fuchsian group G_1 represents an equivalent pair. Then there exist conformal mappings ϕ and ψ with $\phi(I(\gamma_G)) = I(\gamma_{G_1})$, $\psi(E(\gamma_G)) = E(\gamma_{G_1})$, $\phi G\phi^{-1} = \psi G\psi^{-1} = G_1$, and, for every $A \varepsilon G$,

$\phi A\phi^{-1} = B\psi A\psi^{-1}B^{-1}$, B being a fixed element of G_1. Since ψ may be replaced by $B\psi$, we lose no generality in assuming that $B = 1$. The functions $\phi(z)$ and $\psi(z)$ are conformal homeomorphisms between Jordan domains and hence topological on γ_{G_o}. Since $\phi A\phi^{-1} = \psi A\psi^{-1}$ for $A\varepsilon G$, we have that $\phi = \psi$ at the fixed points of A. Therefore $\phi = \psi$ on γ_{G_o} and there exists a homeomorphism C of P onto itself such that $CGC^{-1} = G_1$, $C(z) = \phi(z)$ in $I(\gamma_{G_o})$ and $C(z) = \psi(z)$ in $E(\gamma_{G_o})$.

Using known properties of Ω_m (cf. [1]) and a standard reasoning we verify that $C\Omega_m$ has L_2 derivatives everywhere; so, therefore, does C. Since $\partial C/\partial \bar{z} = 0$ a.e., C is conformal and hence a Möbius transformation.

3. Consider now a fixed closed Riemann surface S_o of genus $g > 1$. The equivalence classes of even coupled pairs $(S, [f], S_o)$ are the points of the Teichmüller space T_g. It is known that T_g has a natural complex-analytic structure and can be represented as a bounded domain in the number space C^{3g-3}; also T_g is homeomorphic to a cell (ct. [2,3] and the reference given there). If $\tau = (\tau_1, \ldots, \tau_{3g-3}) \in T_g$, we denote by $(S_\tau, [f_\tau], S_o)$ any pair represented by τ. There exists a properly discontinuous group Γ_g of holomorphic automorphismis of T_g such that S_{τ_1} is conformally equivalent to S_{τ_2} if and only if τ_1 and τ_2 are equivalent under Γ_g.

Theorem 2. *There exist 2g Möbius transformations $A_j^{(\tau)}$ which depend holomorphically on $\tau \in T_g$, satisfy the normalization conditions:* $A_{2g-1}(0) = 0$, $A_{2g-1}(\infty) = \infty$, $A_{2g}(1) = 1$, $\prod_{j=1}^{g} A_{2j-1}A_{2j}A_{2j-1}^{-1}A_{2j}^{-1} = 1$, *and generate, for each fixed τ, a quasi-Fuchsian group G_τ with $I(\gamma_{G_\tau})/G_\tau$ conformally equivalent to S_τ.*

Holomorphic dependence of $A^{(\tau)}$ on $\tau \in T_g$ means, of course, that $A^{(\tau)}(z) = [a(\tau)\, z + b(\tau)]/[c(\tau)z + d(\tau)]$, where a, b, c, d are holomorphic functions.

Sketch of proof. We may assume that $0 \in T_g$ corresponds to the pair $(S_o, [1], S_o)$. Let G_o be the Fuchsian group (with γ_{G_o} the real axis) representing the odd pair $(S_o, [\iota], \bar{S}_o)$, and let

$\{A_1^{(0)}, \ldots, A_{2g}^{(0)}\}$ be a suitably normalized set of generators of G_o. Every pair $(S_\zeta, [f_\zeta], S_o)$ is equivalent to one of the form $(S_o^m, [1], S_o)$. Let Ω_m be as in the proof of Theorem 1, and set $A_j^{(\zeta)} = \Omega_m A_j^{(0)} \Omega_m^{-1}$. Using Theorem 1 and the properties of Ω_m proved in [1], as well as the definition of the complex analytic structure of T_g (cf. [2]), one verifies that $A_j^{(\zeta)}$ depend only on ζ and not on m, and have the required properties.

Note that $\gamma_G = \Omega_m(\gamma_{G_o})$, so that this curve admits the representation $z = \sigma(t,\tau)$, $-\infty < t < \infty$ where σ depends holomorphically on τ, and $\sigma \longrightarrow \infty$ for $|t| \longrightarrow \infty$.

4. Next, let S_o be as before, and let S_1 denote the surface obtained by removing some fixed point from S_o. The equivalence classes of even coupled pairs $(S,[f],S_1)$ are the points of the Teichmüller space $T_{g,1}$ which is again a complex manifold homeomorphic to a cell and representable as a bounded domain in C^{3g-2}. Using the methods of the proof of Theorem 2 it is not difficult to establish

Theorem 3. <u>$T_{g,1}$ is holomorphically equivalent to the domain $M_{g,1} \subset C^{3g-2}$ defined as follows: $(z,\zeta) = (z, \zeta_1, \ldots \zeta_{3g-3}) \in M_{g,1}$ if and only if $\zeta \in T_g$ and $z \in I(\gamma_{G_\zeta})$.</u>

The results of [2, § 10] can now be restated as

Theorem 4. <u>There exist finitely many meromorphic functions, $F_1(z,\zeta)$, ..., $F_N(z,\zeta)$, in $M_{g,1}$ which, for every fixed ζ, generate the field of automorphic functions in $I(\gamma_{G_\zeta})$ under the group G_ζ, i.e. - the field of meromorphic functions on S_ζ.</u>

These functions uniformize simultaneously all algebraic function fields of genus g, just as the functions $\wp(z,1,\tau)$, $\wp'(z,1,\tau)$, $|z| < \infty$, $\operatorname{Im} \tau > 0$, uniformize all elliptic function fields.

<u>5</u>. Finally let S_o be any open Riemann surface without non-trivial conformal self mapping homotopic to the identity. The Teichmüller space $T(S_o)$, i.e. the space of equivalence classes of even pairs $(S,[f],S_o)$ is a complete metric space (under the Teichmüller distance) but, in general, infinitely dimensional. Nevertheless we may define a continuous complex valued function Φ on $T(S_o)$ to be <u>holomorphic</u> if for every $p_1 = (S_1,[f],S_o) \in T(S_o)$ and every finite sequence $(m_1, \ldots, m_r)$ of Beltrami's differentials on S_1, the mapping of a neighborhood of $0 \in C^r$ into C given by $(\zeta_1, \ldots, \zeta_r) \to p = (S_1^{\zeta_1 m_1 + \ldots + \zeta_r m_r}, [f], S_o) \to \Phi(p)$ is holomorphic. The method of proof of theorem 2 yields

<u>Theorem</u> 5. <u>If $(S_o,[\iota],\bar{S}_o)$ is representable by a Fuchsian group of the first kind, then there exist a finite or infinite sequence of Möbius transformation $\{A_j^{(p)}\}$, depending holomorphically on $p \in T(S_o)$ and such that, for every fixed $q = (S_1,[f],S_o) \in T(S_o)$, the $A_j^{(q)}$ generate a quasi-Fuchsian group G_q with $I(\gamma_{G_q})/G_q$ conformally equivalent to S_1.</u>

Thus there are many holomorphic functions as $T(S_o)$, in particular, enough functions to <u>separate points</u>.

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